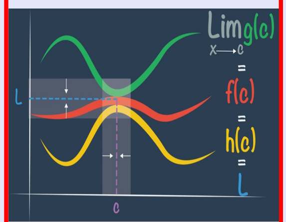
Math 261

Spring 2022

Lecture 20



Class QZ 12

Find all critical numbers $\rightarrow$ $f'(x) = 0$ or $f'(x)$ undefined for

$f(x) = 4x - \tan x$ on $[0, 2\pi)$

$f'(x) = 4 - \sec^2 x$

$f'(x) = 0 \rightarrow 4 - \sec^2 x = 0 \rightarrow \sec^2 x = 4 \rightarrow \cos^2 x = \frac{1}{4}$

$\cos x = \pm\sqrt{\frac{1}{4}}$ $\quad \cos x = \pm \frac{1}{2}$

Ref. Angle 60°

QI $\rightarrow 60^\circ \rightarrow \frac{\pi}{3}$

QII $\rightarrow 120^\circ \rightarrow \frac{2\pi}{3}$

QIII $\rightarrow 240^\circ \rightarrow \frac{4\pi}{3}$

QIV $\rightarrow 300^\circ \rightarrow \frac{5\pi}{3}$

$f'(x) = 4 - \sec^2 x$

$\sec x$ is undefined

where $\cos x = 0$

at $\boxed{\frac{\pi}{2}, \frac{3\pi}{2}}$

Rolle's Theorem

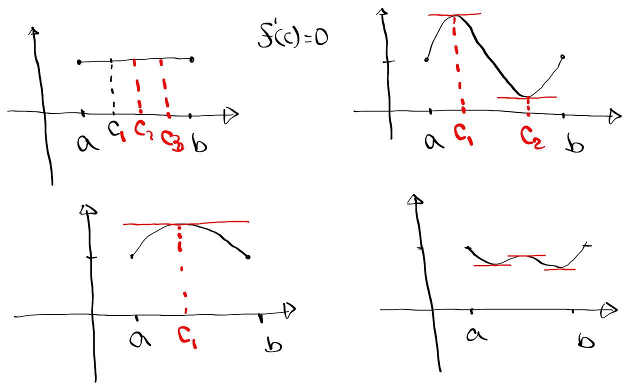
Let $f(x)$ be a function such that

1) It is continuous on $[a,b]$,

2) It is differentiable on (a,b), and

3) $f(a) = f(b)$

then there is a number c in (a,b) such that $f'(c) = 0$



$f(x) = x^3 - x^2 - 6x + 2$, $[0,3]$

Rolle's Thrm

$f(x)$ is cont. on $[0,3]$

$f(x)$ is diff. on $(0,3)$

$f(0) = 2$, $f(3) = 3^3 - 3^2 - 6(3) + 2$
$= 27 - 9 - 18 + 2 = 2$ $\Rightarrow f(0) = f(3)$

Now find all numbers c in $(0,3)$ such that $f'(c) = 0$

$f'(x) = 3x^2 - 2x - 6$

$f'(c) = 0$

$3c^2 - 2c - 6 = 0$

$a = 3 \quad b = -2 \quad c = -6$

$b^2 - 4ac = 4 - 4(3)(-6)$
$= 4 + 72 = 76$

$c = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{2 \pm \sqrt{76}}{6}$

$c = \dfrac{2 + \sqrt{76}}{6} \approx 1.786$ which is in $(0,3)$

$c = \dfrac{2 - \sqrt{76}}{6}$ Not in $(0,3)$.

$f(x) = \sqrt{x} - \frac{1}{3}x \;,\; [0,9]$

$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3} \qquad (0,9)$

$f(0) = 0 = f(9)$

$f'(c) = 0$ on $(0,9)$

$$\frac{1}{2\sqrt{c}} - \frac{1}{3} = 0$$

$$\frac{1}{2\sqrt{c}} = \frac{1}{3}$$

$$2\sqrt{c} = 3$$

$$4c = 9$$

$$c = \frac{9}{4} = \boxed{2.25}$$

Mean-Value theorem:

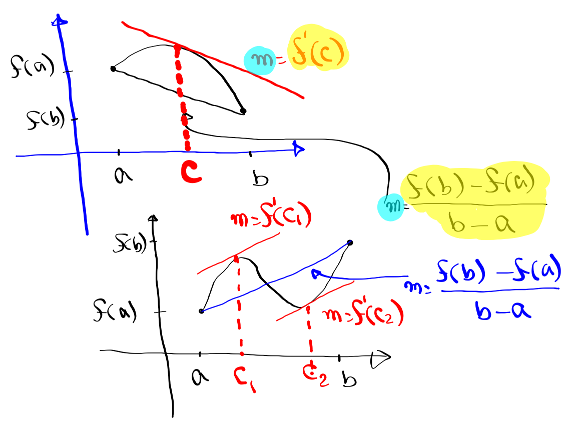
Let $f(x)$ be a function such that

1) It is continuous on $[a,b]$, and

2) It is differentiable on (a,b)

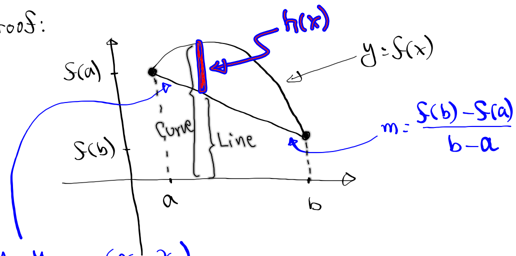
then there is a number c in (a,b)

Such that $\quad f'(c) = \frac{f(b) - f(a)}{b-a}$



April 25, 2022

Proof:



$$y - y_1 = m(x - x_1)$$

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

$$y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

h(x) = Curve − line

$$= f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]$$

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

1) $h(x)$ is the difference of two cont. Function

$\therefore$ it is continuous

2) $h(x)$ is the difference of two diff. Function

$\therefore$ it is differentiable

3) $h(a) = 0$ $\quad$ $h(b) = 0$

So $h(x)$ satisfies all three conditions of the Rolle's Thrm, therefore

$h'(c) = 0$ on (a, b)

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

$$h'(x) = f'(x) - 0 - \frac{f(b)-f(a)}{b-a} \cdot 1$$

$h'(c) = 0$

$$f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Conclusion of MVT.

$f(x) = x^3 - 3x + 2$, $[-2, 2]$

$f(x)$ is a polynomial function

$\Rightarrow$ cont. everywhere

$\Rightarrow$ diff. everywhere

by MVT, there is a number c in $(-2, 2)$

Such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$f'(x) = 3x^2 - 3$ $\qquad 3c^2 - 3 = \dfrac{4 - 0}{2 - (-2)}$

$f(2) = 2^3 - 3(2) + 2 = 4$

$3c^2 - 3 = 1$

$f(-2) = (-2)^3 - 3(-2) + 2 = 0$

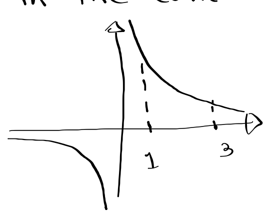
$3c^2 = 4 \qquad c^2 = \dfrac{4}{3}$

$c = \pm \dfrac{2}{\sqrt{3}} = \pm \dfrac{2\sqrt{3}}{3}$

$c \approx \pm 1.155$

$f(x) = \dfrac{1}{x}$, $[1, 3]$

Verify the conditions of MVT, then find all number c in $(1, 3)$ that is in the conclusion of MVT.



$f(x)$ is cont. on $[1, 3]$

" diff " $(1, 3)$

$f'(x) = \dfrac{-1}{x^2}$ $\qquad f(1) = 1$

$f(3) = \dfrac{1}{3}$

$f'(c) = \dfrac{-1}{c^2}$

$\dfrac{-1}{c^2} = \dfrac{\frac{1}{3} - 1}{3 - 1}$ $\qquad \dfrac{-1}{c^2} = \dfrac{-\frac{2}{3}}{2}$

$-\dfrac{2}{3}c^2 = -2 \qquad c^2 = 3$

$c = \pm\sqrt{3}$

$c = \sqrt{3}$

$\sqrt{3} \in (1, 3)$

Side theorems:

If $f'(x)=0$ for all x in (a,b),
then $f(x)$ is constant on (a,b).

If $f'(x) = g'(x)$ for all x in (a,b), then
$f(x) = g(x) + C$ in (a,b) where
C is a constant.

$f'(x)=2x$ $\quad$ $g'(x)=2x$

$f(x)=x^2$ $\quad$ $g(x)=x^2+C$

Suppose $f(1)=10$ and $f'(x)\geq 2$ on $[1,4]$

how small can $f(4)$ possibly be?

at least 16

by MVT

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f'(c) = \frac{f(4)-f(1)}{4-1}$$

$$f'(x) = \frac{f(4)-10}{3} \geq 2$$

$$f(4)-10 \geq 6 \Rightarrow f(4) \geq 16$$

$$f(x) = \frac{2x^2}{x^2-1} \qquad f'(x) = \frac{-4x}{(x^2-1)^2}, \quad f''(x) = \frac{12x^2+4}{(x^2-1)^3}$$

Domain $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$

V.A. $x = \pm 1$

H.A. $y = 2$

x-Int $(0,0)$

y-Int $(0,0)$

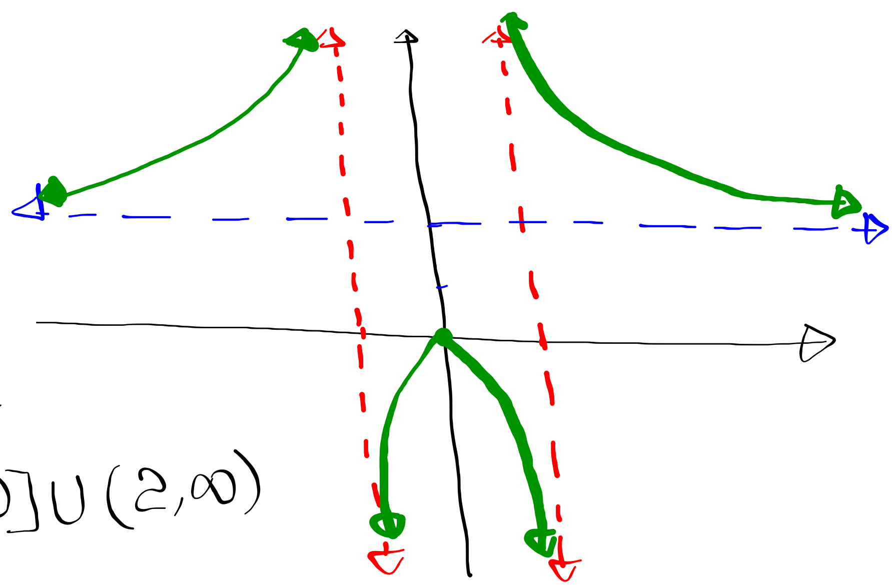
$$f(-x) = \frac{2(-x)^2}{(-x)^2-1} = \frac{2x^2}{x^2-1} = f(x)$$

even function $\Rightarrow$ Symmetric with respect to y-axis.

C.N. $\Rightarrow$ $0, \pm 1$

P.I.P. $\Rightarrow$ ± 1

x	$-\infty$		-1		0		1		∞
$f'(x)$		+	O	+	•	−	O	−	
$f''(x)$		+	O	−		−	O	+	
$f(x)$									



Range

$(-\infty, 0] \cup (2, \infty)$

$f(x) = \dfrac{\cos x}{2+\sin x}$ $[0, 2\pi]$, $f'(x) = -\dfrac{2\sin x + 1}{(2+\sin x)^2}$

Be aware

$2+\sin x > 0$

$-1 \le \sin x \le 1$

$1 \le 2+\sin x \le 3$

$f''(x) = \dfrac{2\cos x(\sin x - 1)}{(2+\sin x)^3}$

1) Domain $(-\infty, \infty)$, $2+\sin x \neq 0$, we focus on $[0, 2\pi]$

2) C.N. $f'(x) = 0$, undefined

$2\sin x + 1 = 0$ $\sin x = \frac{-1}{2}$

Ref. Angle $30° \to \frac{\pi}{6}$ $30°$

QIII $\to \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ $210°$

QIV $\to 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$ $330°$

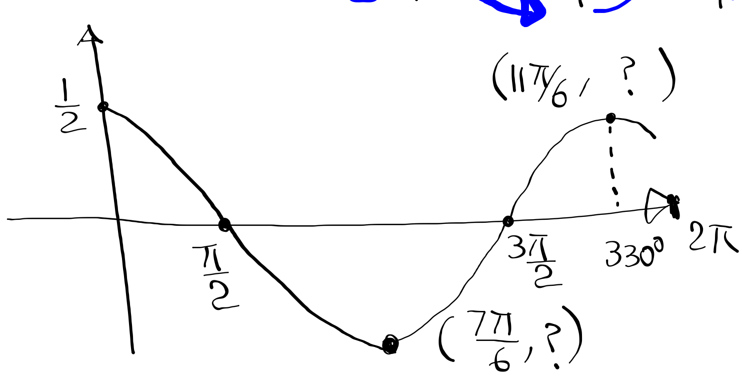
3) P.I.P. $f''(x) = 0$, undefined

$2\cos x(\sin x - 1) = 0$ $\to 90°$

$\cos x = 0 \to x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\to 270°$

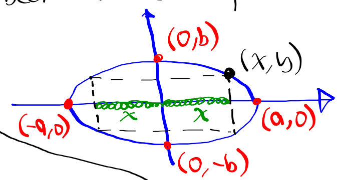
$\sin x - 1 = 0 \to x = \frac{\pi}{2}$

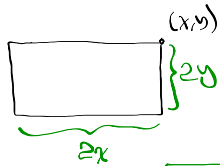
x	0		$\pi/6$ ($30°$)		$\pi/2$ ($90°$)		$7\pi/6$ ($210°$)		$3\pi/2$ ($270°$)		$11\pi/6$ ($330°$)		2π
$f'(x)$		$-$	●	$-$		$-$	●	$+$		$+$	●	$-$	
$f''(x)$		$-$		$-$	●	$+$		$+$	●	$-$		$-$	
$f(x)$		↘		↘		↘		↗		↗		↘	



Find the area of the largest rectangle that can be inscribed in the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



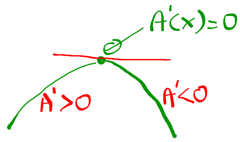


Area $=4xy$

$$b^2x^2+a^2y^2=a^2b^2$$
$$a^2y^2=a^2b^2-b^2x^2$$
$$y^2=\frac{a^2b^2-b^2x^2}{a^2}$$
$$y=\frac{\sqrt{a^2b^2-b^2x^2}}{a}$$

$$A(x)=4x\cdot\frac{\sqrt{a^2b^2-b^2x^2}}{a}$$

$A'(x)$



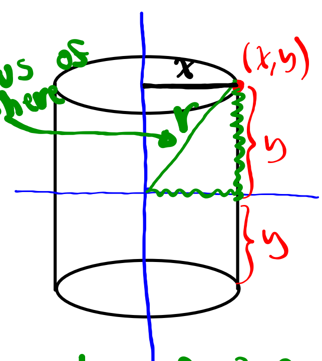
Find $A'(x)$

Solve $A'(x)=0$

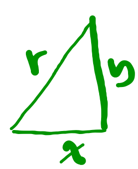
Show

A right circular cylinder is inscribed in a sphere with radius r.

Find the largest possible volume of such cylinder.



$$V=\pi r^2h$$
$$=\pi r^2\cdot 2y$$
$$=\pi x^2\cdot 2y$$
$$V=2\pi x^2y$$
$$V(y)=2\pi y(r^2-y^2)$$



$$x^2+y^2=r^2$$
$$x^2=r^2-y^2$$

$V'(y)$

Solve $V'(y)=0$